AdS/QCD and its Holographic Light-Front Partonic Representation

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Abstract. Starting from the Hamiltonian equation of motion in QCD we find a single variable light-front equation for QCD which determines the eigenspectrum and the light-front wavefunctions of hadrons for general spin and orbital angular momentum. This light-front wave equation is equivalent to the equations of motion which describe the propagation of spin-J modes on anti-de Sitter (AdS) space.

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INTRODUCTION

Quantum Chromodynamics, the Yang-Mills local gauge field theory of $SU(3)_C$ color symmetry, provides a fundamental description of hadronic physics in terms of quark and gluon degrees of freedom. Yet, because of the strongly coupling nature of QCD in the infrared domain, it has been difficult to determine how the constituents appear in the physical spectrum as colorless states, mesons and baryons, or to make precise predictions for hadronic properties outside of the perturbative regime. Thus an important theoretical objective is to find an initial approximation to relativistic bound state problems in QCD which is analytically tractable and which can be systematically improved.

Light-front quantization is the ideal framework to describe the structure of hadrons in terms of their quark and gluon degrees of freedom. The simple vacuum structure in the light-front (LF) allows unambiguous definition of the partonic content of a hadron: partons in a hadronic state are described by light-front wave functions (LFWFs) which encode the hadronic properties. The LFWFs of bound states in QCD are relativistic generalizations of the Schrödinger wavefunctions of atomic physics, but they are determined at fixed light-cone time $\tau = t + z/c$ – the "front form" introduced by Dirac [1] – rather than at fixed ordinary time t.

In this talk, we show that there is an invariant light-front coordinate ζ which allows the separation of the essential dynamics of quark and gluon binding from the kinematical physics of the constituents. The result is a single-variable LF equation for QCD which determines the spectrum and the LFWFs of hadrons for general spin and orbital angular momentum. Our analysis follows from recent developments in light-front QCD [2, 3, 4, 5, 6] which have been inspired by the AdS/CFT correspondence [7] between string states in anti-de Sitter (AdS) space and conformal field theories (CFT) in physical space-time. The use of AdS space and conformal methods in QCD can be motivated from the empirical evidence [8] and theoretical arguments [9] that the QCD cou-

pling $\alpha_s(Q^2)$ has an infrared fixed point at low Q^2. As we have shown recently, there is a remarkable mapping between the description of hadronic modes in AdS space and the Hamiltonian formulation of QCD in physical space-time quantized on the light-front [2]. This procedure allows string modes $\Phi(z)$ in the AdS holographic variable z to be precisely mapped to the LFWFs of hadrons in physical space-time in terms of a specific LF variable ζ which measures the separation of the quark and gluonic constituents within the hadron. This mapping was originally obtained by matching the expression for electromagnetic current matrix elements in AdS space with the corresponding expression for the current matrix element using LF theory in physical space time [3, 4]. More recently we have shown that one obtains a consistent holographic mapping using the matrix elements of the energy-momentum tensor [10], thus providing an important verification of holographic mapping from AdS to physical observables defined on the light front.

A SINGLE-VARIABLE LIGHT-FRONT EQUATION FOR QCD

To a first approximation light-front QCD is formally equivalent to an effective gravity theory on AdS_5. To prove this, we show that the LF Hamiltonian equation of motion of QCD leads to an effective LF wave equation for physical modes $\phi(\zeta)$ which encode the hadronic properties. We compute the hadron mass $\mathcal{M}^2$ from the hadronic matrix element $\langle\psi_H(P')|H_{LF}|\psi_H(P)\rangle = \mathcal{M}_H^2\langle\psi_H(P')|\psi_H(P)\rangle$, where H_{LF} is the Lorentz invariant Hamiltonian $H_{LF} = P_\mu P^\mu = P^- P^+ - \mathbf{P}_\perp^2$ and the state $|\psi_H\rangle$ is an expansion in multi-particle Fock states $|n\rangle$ of the free LF Hamiltonian: $|\psi_H\rangle = \sum_n \psi_{n/H}|n\rangle$. To simplify the discussion we will consider a two-parton hadronic bound state in the limit of massless constituents. We find [2]

$$\begin{aligned}
\mathcal{M}^2 &= \int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \frac{\mathbf{k}_\perp^2}{x(1-x)} |\psi(x,\mathbf{k}_\perp)|^2 + \text{interactions} \\
&= \int_0^1 \frac{dx}{x(1-x)} \int d^2\mathbf{b}_\perp\, \psi^*(x,\mathbf{b}_\perp) \left(-\nabla^2_{\mathbf{b}_{\perp\ell}}\right) \psi(x,\mathbf{b}_\perp) + \text{interactions}. \qquad (1)
\end{aligned}$$

The functional dependence for a given Fock state is given in terms of the invariant mass $\mathcal{M}_n^2 = \left(\sum_{a=1}^n k_a^\mu\right)^2 \to \frac{\mathbf{k}_\perp^2}{x(1-x)}$, the measure of the off-mass shell energy $\mathcal{M}^2 - \mathcal{M}_n^2$. Similarly in impact space the relevant variable is $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$ for a two-parton state. Thus, to first approximation LF dynamics depend only on the boost invariant variable $\mathcal{M}_n$ or ζ and hadronic properties are encoded in the hadronic mode $\phi(\zeta)$: $\psi(x,\mathbf{k}_\perp) \to \phi(\zeta)$. We choose the normalization of the LF mode $\phi(\zeta) = \langle\zeta|\phi\rangle$ with $\int d\zeta\, |\langle\zeta|\phi\rangle|^2 = 1$. Comparing with the LFWF normalization, we find the functional relation: $\frac{|\phi|^2}{\zeta} = \frac{2\pi}{x(1-x)}|\psi(x,\mathbf{b}_\perp)|^2$.

We write the Laplacian operator in circular cylindrical coordinates (ζ,φ), and factor out the angular dependence of the modes in terms of the $SO(2)$ Casimir representation L^2 of orbital angular momentum in the transverse plane: $\phi(\zeta,\varphi) \sim e^{\pm iL\varphi}\phi(\zeta)$. We find

$$\mathcal{M}^2 = \int d\zeta\, \phi^*(\zeta)\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2}\right)\phi(\zeta) + \int d\zeta\, \phi^*(\zeta) U(\zeta) \phi(\zeta), \qquad (2)$$

where all the complexity of the interaction terms in the QCD Lagrangian is summed up in the effective potential $U(\zeta)$. The light-front eigenvalue equation $H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle$ is thus a light-front wave equation for ϕ

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U(\zeta)\right)\phi(\zeta) = \mathcal{M}^2\phi(\zeta), \tag{3}$$

an effective single-variable light-front Schrödinger equation which is relativistic, covariant and analytically tractable. One can readily generalize the equations to allow for the kinetic energy of massive quarks [5].

As the simplest example we consider a bag-like model where the partons are free inside the hadron and the interaction terms will effectively build confinement. The effective potential is a hard wall: $U(\zeta) = 0$ if $\zeta \leq \frac{1}{\Lambda_{\rm QCD}}$ and $U(\zeta) = \infty$ if $\zeta > \frac{1}{\Lambda_{\rm QCD}}$. If $L^2 \geq 0$ the LF Hamiltonian is positive definite $\langle\phi|H_{LF}|\phi\rangle \geq 0$ and thus $\mathcal{M}^2 \geq 0$. If $L^2 < 0$ the LF Hamiltonian is unbounded from below and the particle "falls towards the center". The critical value corresponds to $L = 0$. The mode spectrum follows from the boundary conditions $\phi(\zeta = 1/\Lambda_{\rm QCD}) = 0$, and is given in terms of the roots of Bessel functions: $\mathcal{M}^2_{L,k} = \beta_{L,k}\Lambda_{\rm QCD}$. Since in the conformal limit $U(\zeta) \to 0$, the hard-wall LF model discussed here is equivalent to the AdS/CFT hard wall model of Ref. [11]. Likewise a two-dimensional transverse oscillator with effective potential $U(\zeta) \sim \zeta^2$ is equivalent to the soft-wall model of Ref. [12] which reproduce the usual linear Regge trajectories. Upon the substitution $\zeta \to z$ and $\Phi_J(z) \sim (z/R)^{3/2-J}\phi(z)$ in (3) we find the equation of motion

$$\left[z^2\partial_z^2 - (d-1-2J)z\,\partial_z + z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi_J = 0, \tag{4}$$

describing the propagation of a spin-J mode in AdS$_{d+1}$space. For $d = 4$ the fifth dimensional mass $(\mu R)^2 = -(2-J)^2 + L^2$. The scaling dimensions are $\Delta = 2+L$ independent of J in agreement with the twist scaling dimension of a two parton bound state in QCD.

TRANSITION MATRIX ELEMENTS

Light-Front Holography can be derived by observing the correspondence between matrix elements obtained in AdS/CFT with the corresponding formula using the LF representation [3] . The light-front electromagnetic form factor in impact space [3, 4] can be written as a sum of overlap of light-front wave functions of the $j = 1, 2, \cdots, n-1$ spectator constituents:

$$F(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \sum_q e_q \exp\left(i\mathbf{q}_\perp \cdot \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\right) \left|\psi_n(x_j, \mathbf{b}_{\perp j})\right|^2. \tag{5}$$

The formula is exact if the sum is over all Fock states n. For definiteness we shall consider a two-quark π^+ valence Fock state $|u\bar{d}\rangle$ with charges $e_u = \frac{2}{3}$ and $e_{\bar{d}} = \frac{1}{3}$. For $n = 2$, there are two terms which contribute to the q-sum in (5). Exchanging $x \leftrightarrow 1-x$

in the second integral we find ($e_u + e_{\bar{d}} = 1$)

$$F_{\pi^+}(q^2) = 2\pi \int_0^1 \frac{dx}{x(1-x)} \int \zeta d\zeta \, J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) \left|\psi_{u\bar{d}/\pi}(x,\zeta)\right|^2, \tag{6}$$

where $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$ and $F_\pi^+(q{=}0) = 1$. We now compare this result with the electromagnetic form-factor in AdS space: $F(Q^2) = R^3 \int \frac{dz}{z^3} J(Q^2,z)|\Phi(z)|^2$, where $J(Q^2,z) = zQK_1(zQ)$. Using the integral representation $J(Q^2,z) = \int_0^1 dx J_0\left(\zeta Q\sqrt{\frac{1-x}{x}}\right)$, we can write the AdS electromagnetic form-factor as

$$F(Q^2) = R^3 \int_0^1 dx \int \frac{dz}{z^3} J_0\left(zQ\sqrt{\frac{1-x}{x}}\right) |\Phi(z)|^2. \tag{7}$$

Comparing with the light-front QCD form factor (6) for arbitrary values of Q we find a functional relation between the light-front wave function $\psi(x,\zeta)$ and the AdS wavefunction $\Phi(z)$ consistent with the results found in the previous section from the QCD light-front Hamiltonian eigenvalue equation. We identify the transverse light-front variable ζ, $0 \le \zeta \le \Lambda_{\mathrm{QCD}}$, with the holographic variable z.

CONCLUSION

We have shown that the use of the invariant coordinate ζ in light-front QCD allows the separation of the dynamics of quark and gluon binding from the kinematics of constituent spin and internal orbital angular momentum. The result is a single-variable LF Schrödinger equation which determines the spectrum and LFWFs of hadrons for general spin and orbital angular momentum. This LF wave equation serves as a first approximation to QCD and is equivalent to the equations of motion which describe the propagation of spin-J modes on AdS. This allows us to establish a gauge/gravity correspondence between an effective gravity theory on AdS_5 and light front QCD. Remarkably, the AdS equations correspond to the kinetic energy terms of the partons inside a hadron, whereas the interaction terms build confinement and correspond to the truncation of AdS space. Identical results are found by matching the expression of current matrix elements or the energy-momentum tensor in AdS space with the corresponding expressions using light-front theory in physical space time. This is illustrated in Fig 1, where local operators, such as the current $J^\mu(0)$, are defined at the asymptotic AdS boundary at $z \to 0$ (large circumference) whereas hadronic transition matrix elements like $\langle P'|J^\mu(0)|P\rangle$ probe the hadronic wave function $\Phi(z)$ at a distance $z = 1/Q$, $Q = P' - P$, inside AdS space.

One can systematically improve the holographic approximation by diagonalizing the QCD light-front Hamiltonian on the AdS/QCD basis. The action of the non-diagonal terms in the QCD interaction Hamiltonian generates the form of the higher Fock state structure of hadronic LFWFs. We emphasize, that in contrast with the original AdS/CFT correspondence, the large N_C limit is not required to connect light-front QCD to an effective dual gravity approximation.

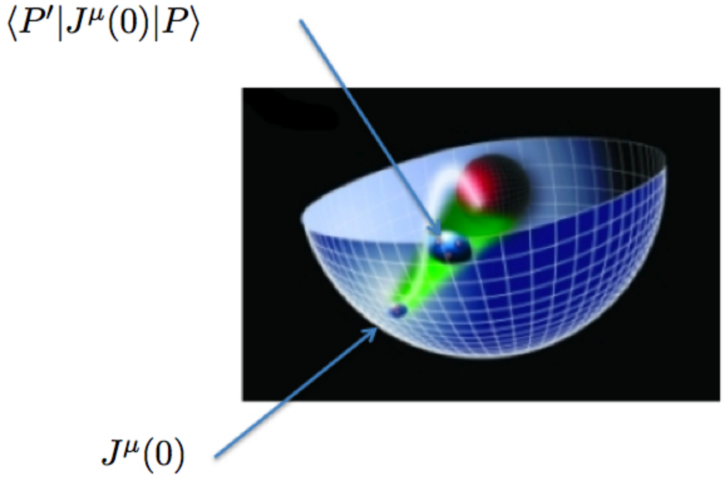

FIGURE 1. AdS representation of light-front QCD. Different values of z correspond to different scales at which the hadron is examined. The AdS boundary at $z \to 0$ (large circumference) corresponds to the $Q \to \infty$, UV, zero separation limit. Local operators are defined at the AdS boundary; hadronic transition matrix elements probe the wave function $\Phi(z)$ (also represented in the figure) at $z = 1/Q$. The inner sphere at large z represents the IR confinement radius. In the figure, a small proton created by a local interpolating operator at the AdS boundary falls into AdS space pulled by the gravitational field up to its larger size allowed by confinement. Due to the warp factor the proton size shrinks by a factor z/R as observed in Minkowski space.

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